# Special lines of quasilattices. II. The case of (2+1)-reducible quasilattices in three dimensions



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# Special lines of quasilattices: II. The case of (2+1)-reducible quasilattices in three dimensions

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**Abstract.** We present a complete classification of special lines from (2+1)-reducible quasilattices, which are periodic along the $c$-axis but only quasiperiodic along the plane perpendicular to it. We also present compatibility relationships between the classes of special lines and those of the special points.

## 1. Introduction

The space group of a Bravais lattice $L$ with point group G is a semidirect product of G and $L$; $\mathscr{G}=\{\{\sigma|l\}|\sigma\in \mathrm{G}$ and $l\in L\}\equiv \mathrm{G}*L$. Let $\mathscr{H}$ be a subgroup of $\mathscr{G}$ and assume that it is a polar point group, i.e. $\mathrm{C}_n$ or $\mathrm{C}_{nv}$ with $n\geqslant 2$. Then, the axis of $\mathscr{H}$ is called a special line (SL) of $L$. We can assume that $\mathscr{H}$ is a maximal polar subgroup of $\mathscr{G}$. Then, the rotational part, H, of $\mathscr{H}$ is called the point group of the SL; H is a polar subgroup of G.

An SL is a particular case of special manifolds of $L$; a special point is associated with a maximal centring subgroup of $\mathscr{G}$ (or G) while a special plane to a mirror of $\mathscr{G}$. The special manifolds are fundamental elements which represent the symmetry of $L$. They are closely related to Wyckoff positions in crystallography (Hahn 1987).

Special manifolds can be defined also for the quasilattice (QL), which plays a similar role in the theory of quasicrystals (see, for example, Janssen 1988) to that of a periodic lattice in conventional crystallography. Special manifolds of a QL are important in the investigation of the local symmetries of a QL (Niizeki 1989a, 1990b). The special manifolds which are equivalent with respect to the 'space group' of the QL form a class. The classes of special points of important QLs in two and three dimensions have been completely classified (Niizeki 1989a, b, 1990a).

In a previous paper (Niizeki 1990b), we established a systematic method for classifying SLs of a QL and, using this method, we have classified SLs of irreducible QLs in two and three dimensions.

There exists an important class of three-dimensional (3D) QLs called (2+1)-reducible QLs, which are periodic along the $c$-axis but only quasiperiodic along the plane perpendicular to the $c$-axis (Janssen 1988, see also Niizeki 1990a). The relevant point groups are $\mathrm{D}_{nh}(n/mm)$ with $n=8$, 10 and 12 (hereafter $n$ is assumed to take these values) and $\mathrm{D}_{5d}(\bar{5}m)$, which reduce the 3D Euclidean space $E_3$ as $E_3=E_2\oplus E_1$; $E_1$ is parallel to the main axis (the $c$-axis) of the point groups. The aim of the present paper is to present a complete classification of SLs (and special planes) of these QLs.

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In section 2, we will summarize the properties of (2+1)-reducible QLs within the range necessary for later arguments (the details can be found, for example, in Niizeki (1990a)). The special lines and special planes of octagonal, decagonal and dodecagonal QLs are classified in section 3 and those of the pentagonal one in section 4. We will discuss, in section 5, other subjects associated with the special manifolds.

## 2. (2+1)-reducible QLs in three dimensions

A QL in $d$ dimensions is a quasiperiodic but discrete set of points obtained by the cut-and-projection method from a periodic lattice, $L_D$, in higher dimensions, where $D$ ($>d$) stands for the dimensions of $L_D$ (Janssen 1988). $E_d$, into which the QL is embedded, has an incommensurate orientation with respect to $L_D$. The projection, $L$, of the whole $L_D$ onto $E_d$ is called a pre-quasilattice (PQL); the QL is a subset of $L$. $L$ is a $Z$-module generated by basis vectors, $e_i$, $i=1, \ldots, D$, which are linearly independent over $Z$; $L=\{\Sigma_i n_i e_i \mid n_i \in Z\}$. $L$ is not discrete in $E_d$ in contrast to a periodic lattice because $D>d$.

The point group, G, of $L$ is formed by all the orthogonal transformations which leave $L$ invariant. The basis vectors give rise to a $D$-dimensional unimodular representation of G. We are interested in the case where G is non-crystallographic.

Let $\mathscr{G}=\mathrm{G} * L$, the semidirect product of G and $L$. Then, $\mathscr{G}$ is the largest subgroup of the Euclidean group of $E_d$ among those which leave $L$ invariant. $\mathscr{G}$ is called a quasi-space group because it is not a discrete subgroup of the Euclidean group. G is embedded into $\mathscr{G}$ and is considered to be a subgroup of $\mathscr{G}$.

There exist three Bravais classes of two-dimensional (2D) PQLs satisfying $D=4$, namely, the $n$-gonal PQL, *Pnmm*. On the other hand, we are interested in (2+1)-reducible 3D PQLs satisfying $D=5$. There exist five Bravais classes of such PQLs, i.e. $Pn/mm$, $F8/mm$ and $P\bar{5}m$ (Janssen 1988). These PQLs are periodic along the $c$-axis; the interlayer distance is denoted by $c$.

The horizontal plane passing the origin is called the basal plane. The basal plane is a lattice plane of a (2+1)-reducible PQL. The lattice points on the basal plane form a 2D PQL; they are distributed densely. Other horizontal lattice planes are translationally equivalent to the 2D PQL. The projection of the 3D PQL onto the basal plane is another 2D PQL, which is identical to the former in the case of a primitive $n$-gonal PQL, $Pn/mm$.

$Pn/mm$ is a direct product of a 2D PQL, *Pnmm*, and one-dimensional (1D) periodic Bravais lattice, $L_1=\{kc \mid k \in Z\}$; $Pn/mm = Pnmm \times L_1$. $Pn/mm$ is a vertical stacking of *Pnmm*. The point group, $\mathrm{C}_{n\mathrm{v}}(nmm)$, of *Pnmm* is embedded into the point group, $\mathrm{D}_{n\mathrm{h}}(n/mm)$, of $Pn/mm$.

$P8/mm$ is a projection onto $E_3$ of a 5D lattice which is an affine distortion of a simple hypercubic lattice in 5D. $F8/mm$ is a sublattice of $P8/mm$ with index 2; it is the projection of the face-centred sublattice of the 5D lattice. The body-centred version, $I8/mm$, belongs to the same Bravais class as that of $F8/mm$. $F8/mm$ is an alternating stacking of two 2D primitive octagonal PQLs ($P8mm$) which are translationally equivalent; the union of the two 2D PQLs is another $P8mm$, which is nothing but a layer of $P8/mm$. The relationship between $P8/mm$ and $F8/mm$ is similar to that between the primitive tetragonal lattice, $P4/mm$, and the face-centred one, $F4/mm$ ($\equiv I4/mm$).

$P\bar{5}m$ is a sublattice of $P10/mm$. It is a periodic stacking of five 2D decagonal PQLs ($P10mm$) which are translationally equivalent. The projection of $P\bar{5}m$ onto the basal plane is identical to a layer of $P10/mm$. The relation between $P10/mm$ and $P\bar{5}m$ is similar to that between the primitive hexagonal lattice, $P6/mm$, and the trigonal lattice, $R\bar{3}m$. $P\bar{5}m$ is obtained, alternatively, as a projection of an affine distortion of the 5D simple hypercubic lattice along [11111]; the orientation of $E_3$ with respect to the 5D lattice is different from the case of $P8/mm$.

In the case of a (2+1)-reducible PQL, every lattice vector parallel to the $c$-axis is a multiple of the vertical basis vector. On the contrary, two horizontal lattice vectors which are parallel can be incommensurate because the PQL is quasiperiodic along the basal plane; a horizontal lattice direction is specified with two lattice vectors (Katz and Duneau 1986).

A lattice line of a PQL passes an infinite number of lattice points. It is parallel to a lattice direction.

Let $L$ be a (2+1)-reducible PQL. Then, its lattice points are special points of the full symmetry. They form class Γ of special points. Every (2+1)-reducible PQL has the second class of special points with full symmetry. These special points are called non-trivial full symmetry points (NTFSPs). Let $x_0$ be a representative of the class of NTFSPs. Then, $L^{\#} = x_0 + L$ $(=\{x_0 + l \mid l \in L\})$ is called the dual lattice to $L$ because $x_0 + L^{\#} = L$; $L^{\#}$ is nothing but the set of all the NTFSPs belonging to the class. The quasi-space-group is common between $L$ and $L^{\#}$. If $L$ has two or more classes of NTFSPs, $L$ has a dual lattice with respect to each class. $F8/mm$, $P10/mm$, $P12/mm$ or $P\bar{5}m$ has only one class of NTFSPs but $P8/mm$ has three.

## 3. The special lines of the octagonal, decagonal and dodecagonal PQLs

Special manifolds of (2+1)-reducible QLs are obtained from those of the relevant PQLs, so that we shall investigate the latter. Since the situation is somewhat different in the case of $P\bar{5}m$, we shall postpone the investigation of this case into the next section. It will help to understand the theory in this section if we refer to the cases of two tetragonal Bravais lattices and that of the hexagonal one because these lattices have similar properties to those of the two octagonal PQLs and the decagonal (or dodecagonal) one, respectively. Some parts of section 3.2 will be summaries of the results of Niizeki (1990b).

### *3.1. Special manifolds of point groups*

Let G be the point group of a (2+1)-reducible PQL, $L$, and let H be a maximal polar subgroup of G. Then, the axis of H is called a special line (SL) of G and H the point group of the SL. Two SLs of G are equivalent if their point groups are conjugate in G. $D_{nh}(n/mm)$ has three inequivalent maximal polar subgroups, $C_{nv}(nmm)$, $C_{2v}(2mm)$ and $C'_{2v}(2mm)$; the latter two are isomorphic to each other. The SL associated with $C_{nv}$ is identical to the $c$-axis while those with the latter two are perpendicular to it.

The SLs associated with the three polar groups are denoted as Λ, Δ and Σ, respectively; Λ is identical to a vertical lattice line but Δ and Σ are to horizontal ones. In the case of $Pn/mm$, we choose Δ so that it is parallel to a horizontal basis vector. Then Σ bisects the angle formed by adjacent Δs. On the other hand, Δ and Σ of $F8/mm$ are chosen so that they agree with the corresponding SLs of $P8/mm$ which has $F8/mm$ as a sublattice.

$D_{nh}$ has three inequivalent mirrors, $\sigma_h$, $\sigma_v$ and $\sigma'_v$; $\sigma_h$ is horizontal but $\sigma_v$ and $\sigma'_v$ are vertical. Accordingly, $D_{nh}$ has three inequivalent special planes; those associated with $\sigma_h$, $\sigma_v$ and $\sigma'_v$ are denoted by $\lambda$, $\delta$ and $\sigma$, respectively. $\lambda$ is identical to the basal lattice plane of the $n$-gonal PQL while $\delta$ and $\sigma$ are to vertical lattice planes. $\delta$ and $\sigma$ are so chosen that they includes $\Delta$ and $\Sigma$, respectively.

An important property of $D_{nh}$ is that each of its three maximal polar subgroups, $C_{nv}$, $C_{2v}$ and $C'_{2v}$, has a companion mirror with which it generates $D_{nh}$.

### 3.2. *General theory of special manifolds of PQLs with point group* $D_{nh}$

Let $x_0$ be a special point of $L$ whose point group is $D_{nh}$ and let H be a maximal polar subgroup of the point group of $x_0$. Then, the line passing $x_0$ and being parallel to the axis of H is an SL of $L$, whose point group is identical to H. Conversely, every SL of $L$ is obtained in this way. Since H is a subgroup of G, every SL of $L$ is parallel to an SL of G and, accordingly, to a high-symmetry lattice line of $L$.

Two SLs are equivalent if their point groups are conjugate to each other as subgroups of the quasi-space group $g = G * L$. In particular, they are translationally equivalent if they are transformed into each other by translations of $L$. Equivalent SLs form a class of SLs.

An SL is classified into type I or II according to whether it passes lattice points of $L$ or not, respectively. A representative of type I SLs passes the origin, so that it coincides with an SL of G; an SL of type I is a high-symmetry lattice line. A class of type I SLs of $L$ is denoted by the same symbol as that for the relevant SL of G. A type II SL passes a NTFSP if its point group is a maximal polar subgroup of $D_{nh}$; this is due to the property of $D_{nh}$ as mentioned at the end of section 3.1.

Every SL of $L$ is parallel or perpendicular to the $c$-axis but is never oblique to it. The $c$-axis is taken usually to be vertical, so that all the SLs of $L$ are classified into four types, Iv, Ih, IIv and IIh, where v or h refers to vertical or horizontal. $\Lambda$ is of type Iv while $\Delta$ and $\Sigma$ are of type Ih. There are no other type I SLs. The remaining problem for the SLs is to enumerate the classes of type II.

Special planes of $L$ are classified in a similar way. Of the three classes, $\lambda$, $\delta$ and $\sigma$, of type I special planes, $\lambda$ is of type Ih but the other two are of type Iv.

Let $X$ be a class of special points of $L$ and let $Y$ be a class of SLs of $L$. Then, it is stated that $X$ and $Y$ are compatible if a special point of $X$ is located on an SL of $Y$. A necessary condition for the compatibility between $X$ and $Y$ is that the point group of $Y$ is a maximal polar subgroup of $X$. Compatibility relationships can be defined with respect to any pair from the three kinds of special manifolds.

Special manifolds are all common between $L$ and its dual $L^{\#}$. Moreover, if $X$ is a special manifold of $L$ (and, accordingly, of $L^{\#}$), its dual, $X^{\#} = x_0 + X$, is also a special manifold of $L$ (and $L^{\#}$), where $x_0$ is a representaitve of $L^{\#}$; the point group is common between $X$ and $X^{\#}$. Let $C$ be a class of special manifolds of $L$ and let $C^{\#} = \{X^{\#} | X \in C\}$ be its dual. Then, the dual to $C^{\#}$ is equal to $C$, i.e. $(C^{\#})^{\#} = C$. $C$ is called self-dual if $C^{\#} = C$. If $C$ is a non-self-dual class of type I special manifolds, $C^{\#}$ is of type II; $C^{\#}$ is, however, of type I as a class of special manifolds of $L^{\#}$. A class of type I SLs is self-dual if and only if an SL of the class passes NTFSPs in $L^{\#}$.

The set of all the classes of special manifolds must be closed with respect to the dual transformation (DT). Compatibility relationships between different kinds of special manifolds must be consistent with the DT (Niizeki 1990b).

A table of the classes of SLs of a PQL must not incur any omission. The table is complete if the compatibility relationships of the classes of SLs with those of special points satisfy the following conditions.

*Completeness condition* Let $X$ be any class of special points and let H be any maximal polar subgroup of the point group of $X$. Then, there exists a class, say $Y$, of SLs such that (i) $Y$ is compatible with $X$ and (ii) H is the point group of $Y$.

### 3.3. *The case of Pn/mm*

Every primitive $n$-gonal PQL has a NTFSP of the form $x_0 = (0, 0, c/2)$. It follows that every class of vertical SLs is self-dual but that of horizontal ones is not. The same is true for special planes. The dual to $\lambda$, a class of type Ih special planes, is a class of type IIh special planes. This class is denoted by $z$. There exists no other classes of horizontal special planes. A class of special points (or horizontal SLs) is compatible with $\lambda$ or $z$ but not both; the classes of special points (or horizontal SLs) are paired into dual pairs, each of which is consistent with the dual pair $\{\lambda, z\}$ of horizontal special planes.

The classes of special points (or horizontal SLs) of $Pn/mm$ are identified with those of $Pnmm$† if they are compatible with $\lambda$. Accordingly, the comptability relationships between these classes of special points and horizontal SLs have been established already (Niizeki 1990b). These compatibility relationships are transferred to those between the duals of these classes. The duals of $\Delta$, $\Sigma$ and $Y$, the classes of horizontal SLs of $Pn/mm$, are denoted by $E$, $F$ and $G$, respectively. The point group of every class of horizontal SLs is $C_{2v}$.

Since $Pn/mm$ is a direct product of $Pnmm$ and a 1D Bravais lattice, there exists a bijection (a one-to-one correspondence) between the set of all the classes of vertical special lines (or planes) of $Pn/mm$ and that of special points (or lines) of $Pnmm$; the projection of a class of vertical special lines (or planes) of $Pn/mm$ onto the basal plane yields a class of special points (or lines) or $Pnmm$ and this projection, $\pi$, is just the bijection.

Let $X$ be a class of special points of $Pnmm$. Then, $\pi^{-1}(X)$, a class of vertical SLs of $Pn/mm$, is denoted by $\bar{X}$. The point group is common between $X$ and $\bar{X}$. $\bar{\Gamma}$ is identical to $\Lambda$ and we shall use the latter symbol in this case. $\bar{X}$ is compatible only with $X$ and $X^{\#}$.

Let $X$ be a class of SLs of $Pnmm$. Then, $\pi^{-1}(X)$, a class of vertical special planes of $Pn/mm$, is denoted by a lower case letter corresponding to the symbol of $X$; more precisely, $\delta = \pi^{-1}(\Delta)$, $\sigma = \pi^{-1}(\Sigma)$ and $y = \pi^{-1}(Y)$, where $\Delta$, $\Sigma$ and $Y$ are classes of SLs of $Pnmm$ ($Y$ is so only for $P8mm$).

The classes of SLs (or special planes) of the three primitive $n$-gonal PQLs are tabulated in table 1 (or 2). A class of SL (or special planes) is denoted with a capital (or a lower case) letter, which is a Greek or Roman letter according to whether the class is type I or II.

### 3.4. *The case of F8/mm*

The classes of SLs of $F8/mm$ are tabulated in table 3($a$) together with their compatibility relationships with the classes of special points. Scrutinizing the compatibility relationships in table 3($a$), we find that the completeness condition in section 3.2 is satisfied.

† Note that the point groups are different between a class of special points of $Pnmm$ and the corresponding class of $Pn/mm$; if the former point group is $C_{n'v}$, the latter one is $D_{n'h} = C_{n'v} \times C_s$.

**Table 1.** Special lines of primitive *n*-gonal PQLs, *Pn/mm*, with $n = 8(a)$, $10(b)$ and $12(c)$.

The first row of each table shows the symbols of the classes of SLs and the second the types. An SL of type I passes lattice points (Γ) but the type II one does not. The symbol v or h attached to I or II shows that the SLs belonging to the class are vertical or horizontal.

The third row shows the point groups. The point groups, $C_{2v}$ and $C'_{2v}$, of the classes of horizontal SLs are inequivalent in $D_{nh}(nmm)$ although they are isomorphic (they are not equivalent to the point group $C_{2v}$ of a class of vertical SLs).

The fourth row in each table shows a representative class of special points which are passed by SLs belonging to each class (for the symbols of the classes of special points, see Niizeki (1990a)).

The compatibility relationships between the classes of SLs and those of special points are not shown for these PQLs. They are obvious for the case of vertical SLs while, for the case of horizontal SLs, they are reduced to those of the relevant 2D PQLs, which are presented in Niizeki (1990b).

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (*a*) *P*8/*mm* | | | | | | | | | | | | |
| Symbol | Λ | $\bar{X}$ | $\bar{C}$ | $\bar{M}$ | $\bar{R}$ | $\bar{O}$ | Δ | Σ | *Y* | *E* | *F* | *G* |
| Type | Iv | IIv | IIv | IIv | IIv | IIv | Ih | Ih | IIh | IIh | IIh | IIh |
| Point group | $C_{8v}$ | $C_{2v}$ | $C_{2v}$ | $C_{4v}$ | $C_{2v}$ | $C_{8v}$ | $C_{2v}$ | $C'_{2v}$ | $C_{2v}$ | $C_{2v}$ | $C'_{2v}$ | $C_{2v}$ |
| Support | Γ | *X* | *C* | *M* | *R* | *O* | Γ | Γ | *O* | *Z* | *Z* | *P* |
| (*b*) *P*10/*mm* | | | | | | | | | | | | |
| Symbol | Λ | $\bar{X}$ | $\bar{C}$ | $\bar{M}$ | $\bar{P}$ | $\bar{P}'$ | Δ | Σ | *E* | *F* | | |
| Type | Iv | IIv | IIv | IIv | IIv | IIv | Ih | Ih | IIh | IIh | | |
| Point group | $C_{10v}$ | $C_{2v}$ | $C_{2v}$ | $C_{2v}$ | $C_{5v}$ | $C_{5v}$ | $C_{2v}$ | $C'_{2v}$ | $C_{2v}$ | $C'_{2v}$ | | |
| Support | Γ | *X* | *C* | *M* | *P* | *P′* | Γ | Γ | *Z* | *Z* | | |
| (*c*) *P*12/*mm* | | | | | | | | | | | | |
| Symbol | Λ | $\bar{X}$ | $\bar{C}$ | $\bar{M}$ | $\bar{T}$ | $\bar{T}'$ | Δ | Σ | *E* | *F* | | |
| Type | Iv | IIv | IIv | IIv | IIv | IIv | Ih | Ih | IIh | IIh | | |
| Point group | $C_{12v}$ | $C_{2v}$ | $C_{2v}$ | $C_{4v}$ | $C_{3v}$ | $C_{3v}$ | $C_{2v}$ | $C'_{2v}$ | $C_{2v}$ | $C'_{2v}$ | | |
| Support | Γ | *X* | *C* | *M* | *T* | *T′* | Γ | Γ | *Z* | *Z* | | |

*Z* is the only class of NTFSPs of *F*8/*mm*. A representative of *Z* is $(0, 0, c)$. Λ, $\bar{O}$, $\bar{M}$, Δ and *Q* are self-dual. On the other hand, $\{\bar{C}, \bar{C}'\}$ and $\{\Sigma, U\}$ are dual pairs. The compatibility relationships in table 3(*a*) are consistent with the DT.

The classes of special planes of *F*8/*mm* are tabulated in table 2(*c*). They are self-dual and *F*8/*mm* has no type II special planes. The class *Q* of SLs is compatible with no classes of special planes because the point group of *Q* includes no mirrors.

## 4. The case of $P\bar{5}m$

The point group $D_{5d}(\bar{5}m)$ of $P\bar{5}m$ has only two inequivalent maximal polar subgroups, $C_{5v}$ and $C_2$; the axis of $C_{5v}$ is vertical but that of $C_2$ horizontal. Moreover, the five mirrors of $D_{5d}$ are all equivalent. Using these facts together with the table of special points of $P\bar{5}m$ as given in Niizeki (1990a), we can enumerate all the classes of SLs. The results are tabulated in table 3(*b*) together with their compatibility relationships with the classes of special points. They are similar to those in the case of the rhombohedral lattice ($R\bar{3}m$).

$P\bar{5}m$ has only one class of special planes, which are of type Iv. The class is compatible with Λ but with neither Σ nor *S*.

**Table 2.** Special planes of $n$-gonal PQLs: ($a$) $P8/mm$ ($b$); $P10/mm$ and $P12/mm$; and ($c$) $F8/mm$.

The third row in each table shows mirrors associated with the classes of special planes. $\sigma_h$, $\sigma_v$ and $\sigma'_v$ stand for a horizontal mirror and two vertical ones; the latter two are inequivalent in $D_{nh}$.

| | | | | | |
|---|---|---|---|---|---|
| ($a$) $P8/mm$ | | | | | |
| Symbol | $\lambda$ | $z$ | $\delta$ | $\sigma$ | $y$ |
| Type | Ih | IIh | Iv | Iv | IIv |
| Mirror | $\sigma_h$ | $\sigma_h$ | $\sigma_v$ | $\sigma'_v$ | $\sigma_v$ |
| Support | $\Gamma$ | $Z$ | $\Gamma$ | $\Gamma$ | $O$ |
| ($b$) $P10/mm$ and $P12/mm$ | | | | | |
| Symbol | $\lambda$ | $z$ | $\delta$ | $\sigma$ | |
| Type | Ih | IIh | Iv | Iv | |
| Mirror | $\sigma_h$ | $\sigma_h$ | $\sigma_v$ | $\sigma'_v$ | |
| Support | $\Gamma$ | $Z$ | $\Gamma$ | $\Gamma$ | |
| ($c$) $F8/mm$ | | | | | |
| Symbol | $\lambda$ | $\delta$ | $\sigma$ | | |
| Type | Ih | Iv | Iv | | |
| Mirror | $\sigma_h$ | $\sigma_v$ | $\sigma'_v$ | | |
| Support | $\Gamma$ | $\Gamma$ | $\Gamma$ | | |

$P\bar{5}m$ has only one class of NTFSPs, whose representative is $(0, 0, 5c/2)$. $\Lambda$ is self-dual with respect to the DT but $\Sigma$ and $S$ form a dual pair.

## 5. Discussions

Every $(2+1)$-reducible PQL is invariant against a 2D scale change along the basal plane (see, for example, Niizeki 1990a). The scale change is called a self-similarity transformation (ST) because it is related to the self-similarity of the relevant QL. The classes of special manifolds, e.g., SLs are divided into multiplets with respect to the ST; the members of a multiplet are permuted cyclically on the ST. In particular, a singlet is formed by a class which is invariant against the ST. The point group is common among the members of a multiplet. The compatibility relationships must be consistent with the multiplet structure (Niizeki 1990b).

The multiplet structures of the classes of the special points of $(2+1)$-reducible PQLs have been established (Niizeki 1990a). We investigate here the multiplet structure of the classes of SLs. The multiplet structure of the classes of vertical (or horizontal) SLs of $Pn/mm$ are simply related by $\pi$ to that of special points (or lines) of $Pnmm$, which has been established in Niizeki (1990a) (or Niizeki (1990b)). Therefore, we need not consider the case of $Pn/mm$. $F8/mm$ has one doublet, $\{\tilde{C}, \tilde{C}'\}$. Other classes than $\tilde{C}$ and $\tilde{C}'$ form all singlets. On the other hand, all the three classes of SLs of $P\bar{5}m$ form singlets.

A vertical SL of a PQL is a projection onto $E_3$ of an SL of the relevant 5D periodic lattice but a horizontal one is that of its special plane (cf Niizeki 1990b). On the other hand, a horizontal (or vertical) special plane is a projection of a special 4D (or 3D) hyperplane of the 5D lattice.

The local structures associated with special points of $(2+1)$-reducible QLs have been investigated in Niizeki (1990a). The local structures along an SL is strongly

**Table 3.** The classes of special lines of (*a*) $F8/mm$ and (*b*) $P\bar{5}m$ and their compatibility relationships with the classes of special points.

The latter half separated by a horizontal line in each table shows the compatibility relationships. The first column shows the classes of special points. An asterisk means that the relevant class of SLs is compatible with that of special points.

A general point of a representative SL of $\Delta$, a class of the horizontal SLs of $F8/mm$, is indexed as $[xy0\bar{y}0]$, where the index scheme in Niizeki (1990a) is used. Similar parametrizations of representative SLs of $\Sigma$, $U$ and $Q$ of $F8/mm$ are given by $[yxxy0]$, $[yxxy1]$ and $[xyh\bar{y}h]$ with $h=1/2$, respectively, while those of $\Sigma$ and $S$ of $P\bar{5}m$ by $[0yx\bar{x}\bar{y}]$ and $x(\mathrm{I})+[0yx\bar{x}\bar{y}]$ with $x(I)=[11111]/2$ being a representative special point in the class $I$.

| (*a*) $F8/mm$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Symbol | $\Lambda$ | $\bar{C}$ | $\bar{C}'$ | $\bar{M}$ | $\bar{O}$ | $\Delta$ | $\Sigma$ | $U$ | $Q$ |
| Type | Iv | IIv | IIv | IIv | IIv | Ih | Ih | IIh | IIh |
| Point group | $C_{8v}$ | $C_{2v}$ | $C_{2v}$ | $C_{2v}$ | $C_{4v}$ | $C_{2v}$ | $C'_{2v}$ | $C'_{2v}$ | $C_2$ |
| $\Gamma$ | * | | | | | * | * | | |
| $Z$ | * | | | | | * | | * | |
| $C$ | | * | | | | | * | * | |
| $C'$ | | | * | | | | * | * | |
| $M$ | | | | * | | * | | | |
| $O$ | | | | | * | | * | * | |
| $P$ | | | | | * | | | | * |
| $N$ | | | | * | | | | | * |
| $W$ | | | | | | | | | * |
| $S$ | | | | | | | | | * |

| (*b*) $P\bar{5}m$ | | | |
|---|---|---|---|
| Symbol | $\Lambda$ | $\Sigma$ | $S$ |
| Type | Iv | Ih | IIh |
| Point group | $C_{5v}$ | $C_2$ | $C_2$ |
| $\Gamma$ | * | * | |
| $X$ | | | * |
| $C$ | | * | |
| $M$ | | * | |
| $O$ | | | * |
| $P$ | | | * |
| $D$ | | * | |
| $I$ | * | | * |

dependent on the compatibility relations of the relevant class of SLs with the classes of special points (Niizeki 1990b). Therefore, the compatibility relationships between classes of special manifolds are of fundamental importance. The compatibility relationships with respect to SLs of a reciprocal PQL are also important in understanding the change of the 'quasi-dispersion-relation' of an electron in a quasicrystal along an SL in the reciprocal space (Niizeki and Akamatsu 1990).

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